

# THE STRUCTURE OF PERFECT FIELDS

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**ABSTRACT:** This paper builds fundamental perfect fields of positive characteristic and shows the structure of perfect fields that a field of positive characteristic is a perfect field if and only if it is an algebraic extension of a fundamental perfect field.

## 1 Introduction

It has long been known that perfect fields are significant because Galois theory over these fields becomes simpler, since the separable condition (the most important condition of Galois extensions) of algebraic extensions over perfect fields are automatically satisfied. Recall that a field  $k$  is said to be a *perfect field* if any one of the following equivalent conditions holds:

- (a) Every irreducible polynomial over  $k$  is separable.
- (b) Every finite-degree extension of  $k$  is separable.
- (c) Every algebraic extension of  $k$  is separable.

One showed that any field of characteristic 0 is a perfect field. And any field  $k$  of characteristic  $p > 0$  is a perfect field if and only if the Frobenius endomorphism  $x \mapsto x^p$  is an automorphism of  $k$ ; i.e., every element of  $k$  is a  $p$ -th power (see e.g. [1, 2, 3, 4]). In this paper, we show the structure of perfect fields of positive characteristic.

Let  $p$  be a positive prime and  $X = \{x_i\}_{i \in I}$  a set of independent variables. Denote by  $\mathbb{Z}_p$  the field of congruence classes modulo  $p$  and  $\mathbb{Z}_p(X)$  the field of all rational functions in  $X$ . For any  $u \in \mathbb{Z}_p(X)$  and any  $n \geq 0$ , we define  $\sqrt[n]{u}$  to be the unique root of  $y^{p^n} - u \in \mathbb{Z}_p(X)[y]$  in an algebraic closure  $\overline{\mathbb{Z}_p(X)}$  of  $\mathbb{Z}_p(X)$ . Set  $\sqrt[n]{X} = \{\sqrt[n]{x_i}\}_{i \in I}$  for all  $n \geq 0$ . It is clear that  $\mathbb{Z}_p(\sqrt[n]{X}) \subseteq \mathbb{Z}_p(\sqrt[n+1]{X})$  for all  $n \geq 0$ . Then we prove that  $F_p(I) = \bigcup_{n \geq 0} \mathbb{Z}_p(\sqrt[n]{X})$  is a perfect closure of  $\mathbb{Z}_p(X)$  (see Proposition 2.2). Note that we assign  $F_p(I) = \mathbb{Z}_p$  to the case that  $I = \emptyset$ , and call a field  $k$  of characteristic  $p > 0$  is a *fundamental perfect field* if  $k \cong F_p(I)$  for some  $I$ .

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As one might expect, we obtain the following result for the structure of perfect fields of positive characteristic.

**Main theorem** (see Theorem 2.3). *A field of positive characteristic is a perfect field if and only if it is an algebraic extension of a fundamental perfect field.*

The remainder of the paper (Section 2) is devoted to the discussion of fundamental perfect fields (see Proposition 2.2), and proves the main theorem. Moreover, we obtain other interesting information on maximal fundamental perfect subfields of perfect fields (see Corollary 2.4 and Proposition 2.5).

## 2 Fundamental perfect fields

In this section, we will build fundamental perfect fields of positive characteristic and show the structure of perfect fields.

Recall that a *perfect closure*  $F$  of a field  $E$  is a smallest perfect extension of  $E$  in an algebraic closure  $\overline{E}$  of  $E$ ; i.e.,  $F \subseteq F'$  for all  $F'$  perfect and  $E \subseteq F' \subseteq \overline{E}$ . Let  $p$  be a positive prime. Denote by  $\mathbb{Z}_p$  the field of congruence classes modulo  $p$ . And throughout this paper, we always consider  $\mathbb{Z}_p$  as a subfield of fields of characteristic  $p$ .

**Remark 2.1.** Let  $H$  be a field of characteristic  $p$  and  $a \in H$ . Then for any  $n \geq 0$ , the polynomial  $x^{p^n} - a$  has a unique root in an algebraic closure of  $H$  and denote by  $\sqrt[p^n]{a}$  this unique root. And if  $H$  is perfect then  $\sqrt[p^n]{a} \in H$ .

Let  $X = \{x_i\}_{i \in I}$  be a set of independent variables and assume that  $\mathbb{Z}_p(X)$  is the field of all rational functions in  $X$  over  $\mathbb{Z}_p$ . For any  $u \in \mathbb{Z}_p(X)$  and any  $n \geq 0$ , denote by  $\sqrt[p^n]{u}$  the root of  $y^{p^n} - u$  in an algebraic closure  $\overline{\mathbb{Z}_p(X)}$  of  $\mathbb{Z}_p(X)$ . Set  $\sqrt[p^n]{X} = \{\sqrt[p^n]{x_i}\}_{i \in I}$  for all  $n \geq 0$ . It is clear that  $\sqrt[p^n]{X} = (\sqrt[p^{n+1}]{X})^p = \{(\sqrt[p^{n+1}]{x_i})^p\}_{i \in I}$  and  $\mathbb{Z}_p(\sqrt[p^n]{X}) \subseteq \mathbb{Z}_p(\sqrt[p^{n+1}]{X})$  for all  $n \geq 0$ . Consequently,  $F_p(I) = \bigcup_{n \geq 0} \mathbb{Z}_p(\sqrt[p^n]{X})$  is a subfield of  $\overline{\mathbb{Z}_p(X)}$ . Now, we assign  $F_p(I) = \mathbb{Z}_p$  to the case that  $I = \emptyset$ . Note that the structure of  $F_p(I)$  only depends on the cardinality of  $I$ ; i.e., if  $|I| = |J|$  then  $F_p(I) \cong F_p(J)$ .

Then we have the following result.

**Proposition 2.2.**  $F_p(I)$  is a perfect closure of  $\mathbb{Z}_p(X)$ .

*Proof.* Let  $a$  be an element of  $F_p(I)$ . There exists  $n$  such that  $a \in \mathbb{Z}_p(\sqrt[p^n]{X})$ . Hence

$$a = \frac{g(\sqrt[p^n]{x_1}, \dots, \sqrt[p^n]{x_d})}{h(\sqrt[p^n]{x_1}, \dots, \sqrt[p^n]{x_d})}$$

for  $g(y_1, \dots, y_d)$ ;  $h(y_1, \dots, y_d)$  are two polynomials of the polynomial ring  $\mathbb{Z}_p[y_1, \dots, y_d]$  in  $y_1, \dots, y_d$  over  $\mathbb{Z}_p$ . Since  $\mathbb{Z}_p$  is perfect, it follows that

$$g(y_1^p, \dots, y_d^p) = u(y_1, \dots, y_d)^p \quad \text{and} \quad h(y_1^p, \dots, y_d^p) = v(y_1, \dots, y_d)^p$$

for  $u(y_1, \dots, y_d); v(y_1, \dots, y_d) \in \mathbb{Z}_p[y_1, \dots, y_d]$ . Consequently, we get

$$a = \frac{g(\sqrt[p^n]{x_1}, \dots, \sqrt[p^n]{x_d})}{h(\sqrt[p^n]{x_1}, \dots, \sqrt[p^n]{x_d})} = \frac{g((\sqrt[p^{n+1}]{x_1})^p, \dots, (\sqrt[p^{n+1}]{x_d})^p)}{h((\sqrt[p^{n+1}]{x_1})^p, \dots, (\sqrt[p^{n+1}]{x_d})^p)} = \frac{u(\sqrt[p^{n+1}]{x_1}, \dots, \sqrt[p^{n+1}]{x_d})^p}{v(\sqrt[p^{n+1}]{x_1}, \dots, \sqrt[p^{n+1}]{x_d})^p}.$$

Hence  $a = b^p$  for  $b = \frac{u(\sqrt[p^{n+1}]{x_1}, \dots, \sqrt[p^{n+1}]{x_d})}{v(\sqrt[p^{n+1}]{x_1}, \dots, \sqrt[p^{n+1}]{x_d})} \in \mathbb{Z}_p(\sqrt[p^{n+1}]{X}) \subseteq F_p(I)$ . So  $F_p(I)$  is perfect.

Now, assume that  $H$  is a perfect field and  $\mathbb{Z}_p(X) \subseteq H \subseteq \overline{\mathbb{Z}_p(X)}$ . Since  $X \subseteq H$ , it follows that  $\sqrt[p^n]{X} \subseteq H$  for all  $n \geq 0$  by Remark 2.1. Hence  $F_p(I) \subseteq H$ . Thus,  $F_p(I)$  is a perfect closure of  $\mathbb{Z}_p(X)$ .  $\square$

We call a field  $k$  of characteristic  $p > 0$  is a *fundamental perfect field* if  $k \cong F_p(I)$  for some  $I$ . Then we obtain the following interesting result.

**Theorem 2.3.** *A field of positive characteristic is a perfect field if and only if it is an algebraic extension of a fundamental perfect field.*

*Proof.* Let  $E$  be a perfect field of positive characteristic  $p$ . Assume that  $S = \{s_i\}_{i \in I}$  is a transcendence base of  $E$  over  $\mathbb{Z}_p$ . Then it is easily seen that  $E$  is algebraic over  $\mathbb{Z}_p(S)$ . Let  $X = \{x_i\}_{i \in I}$  be a set of independent variables. Since  $S = \{s_i\}_{i \in I}$  is algebraically independent over  $\mathbb{Z}_p$ , it follows that  $\mathbb{Z}_p(S) \cong \mathbb{Z}_p(X)$ . Let  $F$  be a perfect closure of  $\mathbb{Z}_p(S)$ . Since  $E$  is a perfect field, we can consider  $F \subseteq E$ . Remember that  $F_p(I)$  is a perfect closure of  $\mathbb{Z}_p(X)$  by Proposition 2.2 and  $\mathbb{Z}_p(S) \cong \mathbb{Z}_p(X)$ ,  $F \cong F_p(I)$  by [1]. So  $F$  is a fundamental perfect field. And since  $E$  is algebraic over  $\mathbb{Z}_p(S)$ ,  $E$  is algebraic over  $F$ . Conversely, if  $K$  is an algebraic extension of a fundamental perfect field  $k$  then  $K$  is perfect. Indeed, for any algebraic extension  $H$  of  $K$ ,  $H$  is an algebraic extension of  $k$ . Since  $k$  is perfect,  $H$  is separable over  $k$ . Hence  $H$  is separable over  $K$ . Thus,  $K$  is perfect. The proof is complete.  $\square$

A subfield  $F$  of a perfect field  $P$  is called a *maximal fundamental perfect subfield* of  $P$  if  $F$  is a fundamental perfect field and  $P$  is an algebraic extension of  $F$ .

Then from the proof of Theorem 2.3, we find the following corollary.

**Corollary 2.4.** *Let  $P$  be a perfect field of positive characteristic  $p$ . Then the following statements hold.*

- (i)  *$Q$  is a maximal fundamental perfect subfield of  $P$  if and only if there exists a transcendence base  $S = \{s_i\}_{i \in I}$  of  $P$  over  $\mathbb{Z}_p$  such that  $Q$  is the perfect closure of  $\mathbb{Z}_p(S)$  in  $P$ . In this case,  $Q \cong F_p(I)$ .*
- (ii) *The maximal fundamental perfect subfields of  $P$  are isomorphic.*
- (iii) *The structure of maximal fundamental perfect subfields of  $P$  only depends on the cardinality of transcendence bases of  $P$  over  $\mathbb{Z}_p$ .*

*Proof.*  $Q$  is a maximal fundamental perfect subfield of  $P$  if and only if  $Q \cong F_p(I)$  for some  $I$  and  $P$  is an algebraic extension of  $Q$ . On the one hand,  $Q \cong F_p(I)$  if and only if there exists a transcendence base  $S = \{s_i\}_{i \in I}$  of  $Q$  over  $\mathbb{Z}_p$  such that  $Q$  is the perfect closure of  $\mathbb{Z}_p(S)$  in  $P$  by the proof of Theorem 2.3. On the other hand,  $P$  is an algebraic extension of  $Q$  if and only if  $S = \{s_i\}_{i \in I}$  is a transcendence base of  $P$  over  $\mathbb{Z}_p$ . So we get (i). Since if  $|I| = |J|$  then  $F_p(I) \cong F_p(J)$ , hence from the proof of (i) and remember that the cardinalities of transcendence bases of  $P$  over  $\mathbb{Z}_p$  are the same, we have (ii). From (i) and the proof of (ii) we get (iii).  $\square$

Finally, we would like to give the following proposition for perfect closures of fields.

**Proposition 2.5.** *Let  $E$  be a field of positive characteristic  $p$  and  $S = \{s_i\}_{i \in I}$  a transcendence base of  $E$  over  $\mathbb{Z}_p$ . Let  $V = \{v_j\}_{j \in J}$  be a base for the vector space  $E$  over  $\mathbb{Z}_p(S)$  and  $Q$  a perfect closure of  $\mathbb{Z}_p(S)$  in an algebraic closure  $\overline{E}$  of  $E$ . Then  $Q(V)$  is a perfect closure of  $E$ , and  $E$  is perfect if and only if  $Q \subseteq E$ .*

*Proof.* Since  $V$  is algebraic over  $\mathbb{Z}_p(S)$ ,  $Q(V)$  algebraic over  $Q$ . Hence  $Q(V)$  is perfect. Now, assume that  $E \subseteq F \subseteq \overline{E}$  is perfect. Then  $Q \subseteq F$  and hence  $Q(V) \subseteq F$ . So  $Q(V)$  is a perfect closure of  $E$ . From this it follows that  $E$  is perfect if and only if  $E = Q(V)$ . This is equivalent to  $Q \subseteq E$ .  $\square$

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